Newbattle Community High School Maths Department

Intermediate 2 Maths Revision Notes

Units 1, 2 and 3

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Our **Values for Life**

RESPONSIBILITY – PERSEVERANCE – FOCUS – TRUST – RESPECT – HUMOUR

Use this booklet to demonstrate **responsibility** and **perseverance** by practising working independently like you will have to in an exam. This means that when you get stuck on a question, don't just leave the question blank, don't give up, and don't sit there doing nothing until your teacher manages to get to you.

Instead get in the habit of turning to this booklet to refresh your memory.

- If you have forgotten how to do a method, **examples** are given.
- If you have forgotten what a word means, use the **index** to look it up.

This booklet is for:

- Students doing an Intermediate 2 maths course, including Unit 3.

This booklet contains:

- The most important facts you need to memorise for Intermediate 2 maths.
- Examples on how to do the most common questions in each topic.
- Definitions of the key words you need to know.

Use this booklet:

- To refresh your memory of the method you were taught in class when you are stuck on a homework question or a past paper question.
- To memorise key facts when revising for the exam.

The key to revising for a maths exam is to do questions, not to read notes. **As well as using this booklet, you should also:**

- Revise by working through exercises on topics you need more practice on – such as revision booklets, textbooks, or other exercises suggested by your teacher.
- Work through Past papers
- Ask your teacher when you come across a question you cannot answer
- Check the resources online at www.newbattle.org.uk/Departments/Maths/int2.html

As you get closer to the exam you should be aiming to look at this booklet less and less.

Contents

Unit 1

Unit 1 Outcome 1 – Percentages

Percentages

In Intermediate 2 percentage questions, you will always be asked to increase or decrease an amount by a percentage – this will usually be either **compound interest**, or **appreciation** or **depreciation**.

For every question, there is a longer way and a quicker way to do it. Use the one you are happiest with.

Percentage	Longer way	Quicker way
3% increase	0.03×..... Then add answer on	*[100% + 3% = 103%]* 1.03 × .....
3% decrease	0.03×..... Then take answer away	*[100% - 3% = 97%]* 0.97 × .....
2.4% increase	0.024×..... Then add answer on	*[100% + 2.4% = 102.4%]* 1.024 × .....
15% decrease	0.15×..... Then take answer away	*[100% - 15% = 85%]* 0.85 × .....
4.5% decrease	0.045×..... Then take answer away	*[100% - 4.5% = 95.5%]* 0.955 × .....

Compound Interest

Compound interest is an example of **appreciation** (i.e. the amount in the account is always going <u>up</u>), so you always add the amount on each time.

<u>Example</u>

How much compound interest will you receive in 3 years on a balance of £2500 in a savings account that pays 2.4% interest per annum?

Longer method	Quicker method
Year 1: 0.024×2500 = £60 2500 + 60 = £2560	**Year 1:** 2500×1.024=2560
Year 2: 0.024×2560 = £61.44 2560 + 61.44 = £2621.44	**Year 2:** 2560×1.024=2621.44
Year 3: 0.024×2621.44 = £62.91 2621.44 + 62.91 = £2684.35	**Year 3:** 2621.44×1.024=2684.35 *[you could also use even quicker method 2500×1.024^3]*
Interest: 2684.35 – 2560 = <u>£184.35</u>	**Interest:** 2684.35 – 2560 = <u>£184.35</u>

Appreciation and Depreciation

Definition: Appreciation means an increase in value.
Definition: Depreciation means a decrease in value.

Example

Peterhead has a population of 30 000 at the end of 2001. Its population depreciates by 15% per year. What is its population at the end of 2003? Round your answer to 3 significant figures.

Solution

Depreciation means decrease, so we will be taking away.
End of 2001 to end of 2003 is two years so need to repeat 2 times.
100% – 15% = 85%, so we use 0.85 in the quicker method.

Longer method	**Quicker method**
Year 1: $0.15\times30000 = 4500$ $30000 - 4500 = 25500$	**Year 1:** $0.85\times30000=25500$
Year 2: $0.15\times25500 = 3825$ $25500 - 3825 = 21675$	**Year 2:** $0.85\times25500=21675$ *[you could also use even quicker method 30000×0.85^2]*
Answer: 21700 (3s.f.)	**Answer:** 21700 (3s.f.)

Example 2 *where you have to find the percentage first*

A house cost £240 000. One year later its value has appreciated to £250800.

a) Find the rate of appreciation (1 mark)
b) If the house continues to appreciate at this rate, what will its value be after another 4 years?

Solution

a) The increase is 250800–240000=£10 800

As a percentage of the original value (£240 000), this is
$10800\div240000\times100 =$ **4.5%**

b) Using the quicker method:

Appreciation means increase. 100% + 4.5% = 104.5%, so use 1.045
Year 1: $1.045 \times 250800 = 262086$
Year 2: $1.045 \times 262086 = 273879.87$
Year 3: $1.045 \times 273879.87 = 286204.46$
Year 4: $1.045 \times 286204.46 = 299083.67$

[even quicker method: 250800×1.045^4]

Answer: After a further four years the house is worth **£299 083.67**

Unit 1 Outcome 2 – Volumes of Solids

Rounding to Significant Figures

Example 1

446.586 → Rounded to 1 significant figure is **400**
→ Rounded to 2 significant figures is **450**
→ Rounded to 3 significant figures is **447**
→ Rounded to 4 significant figures is **446.6**

Example 2

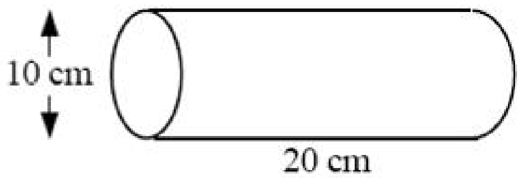
0.00567 → Rounded to 1 significant figure is **0.006**
→ Rounded to 2 significant figures is **0.0057**

Volume of a Cylinder

This formula is given on the exam paper
Volume of a Cylinder: $V = \pi r^2 h$

Example

Work out the volume of this cylinder. Round your answer to 2 significant figures.



Solution

Diameter is 10cm so radius is 5cm

$V = \pi r^2 h$

$V = \pi \times 5 \times 5 \times 20$

$V = 1570.796327....$

$V = 1600\text{cm}^3$ (2s.f.)

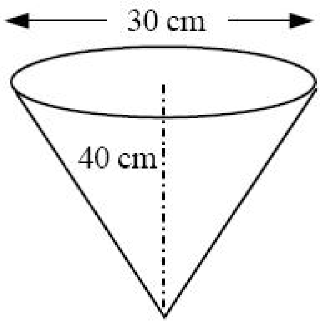
Volume of a Cone

This formula is given on the exam paper
Volume of a Cone: $V = \frac{1}{3}\pi r^2 h$

The height has to be the perpendicular height (the one that goes straight up) and not the diagonal height.

Example

Calculate the volume of this cone. Round your answer to 3 significant figures.



Solution

Diameter is 30cm so radius is 15cm

$V = \frac{1}{3}\pi r^2 h$

$V = \pi \times 15 \times 15 \times 40 \div 3$ (or $1 \div 3 \times \pi \times 15 \times 15 \times 40$

$V = 9424.777961....$

$\underline{V = 9420\text{cm}^3 \text{ (3s.f.)}}$

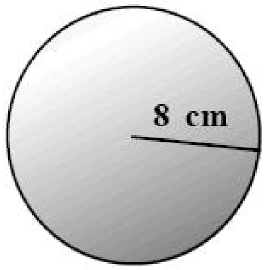
Volume of a Sphere

This formula is <u>given</u> on the exam paper
Volume of a Sphere: $V = \frac{4}{3}\pi r^3$

Definition: A <u>**hemisphere**</u> is half a sphere.

Example 1

Calculate the volume of this sphere. Round your answer to 1 significant figure.



Solution

Radius is 8cm

$V = \frac{4}{3}\pi r^3$

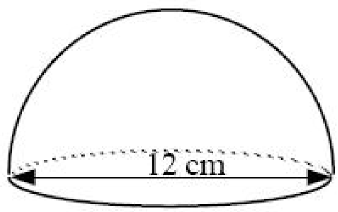
$V = \pi \times 8 \times 8 \times 8 \div 3 \times 4$ (or $4 \div 3 \times \pi \times 8 \times 8 \times 8$)

$V = 2144.660585....$

$\underline{V = 2000\text{cm}^3 \text{ (1s.f.)}}$

Example 2

Calculate the volume of this hemisphere.
Round your answer to 4 significant figures.



Solution

Diameter is 12cm so radius is 6cm

$V = \frac{4}{3}\pi r^3 \div 2$

$V = \pi \times 6 \times 6 \times 6 \div 3 \times 4 \div 2$ (or $4 \div 3 \times \pi \times 8 \times 8 \times 8 \div 2$)

$V = 452.3893421....$

$\underline{V = 452.4\text{cm}^3 \text{ (1s.f.)}}$

Volume of a Prism

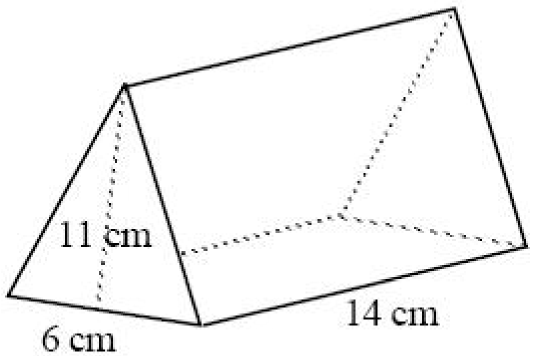
Definition: A **prism** is a 3d solid with a uniform cross-section. In everyday language, this means that is the same shape all the way along.

Definition: The **cross-section** is the shape at either end (and in the middle) of a prism.

This formula is <u>NOT</u> given on the exam paper	
Volume of a prism:	V = Area of cross-section $\times$ length $V = AL$

Example:

Find the volume of this prism



Solution:

Step 1: Work out the area of the cross-section

In this shape, the cross-section is a triangle. The formula for the area of a triangle is $\frac{1}{2}bh$.

$$
\begin{aligned}
A_{triangle} &= \tfrac{1}{2}bh \\
&= 11 \times 6 \div 2 \\
&= 33\text{cm}^2
\end{aligned}
$$

Step 2: Use the formula to find the volume

$$
\begin{aligned}
V &= AL \\
&= 33 \times 14 \\
&= \underline{462\text{cm}^3}
\end{aligned}
$$

Unit 1 Outcome 3 – Straight Line Graphs

The gradient between two points

Definition: the **gradient** of a line is its steepness. It measures how much the line goes up (or down) for every one square you move along. The basic definition of gradient is $\text{gradient} = \frac{\text{up}}{\text{along}} = \frac{\text{vertical}}{\text{horizontal}}$.

A positive gradient means the line slopes upwards, a negative gradient means the line slopes downwards.

This formula is NOT given on the exam paper

Gradient between two points (x_1, y_1) and (x_2, y_2): $m = \frac{y_2 - y_1}{x_2 - x_1}$

Example 1 (two coordinate points)

Find the gradient between the points (–2,5) and (1,4)

Solution:

Step 1 – label the coordinates: $(\underset{x_1}{-2}, \underset{y_1}{5})$ $(\underset{x_2}{1}, \underset{y_2}{4})$

Step 2 – put into the formula:

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

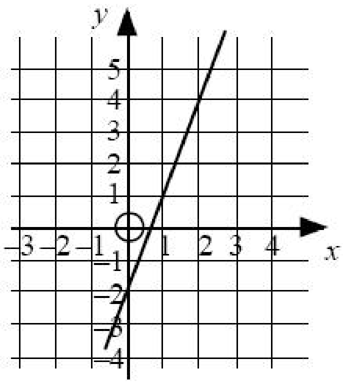
$$m = \frac{4-5}{1--2}$$

$$= \frac{-1}{3}$$

Answer: $m = -\frac{1}{3}$

Example 2 (from a diagram)

Find the gradient of this straight line



Solution:

Step 1 – identify any two coordinates on the line:

$(0,-2)$ $(1,1)$

Step 2 – label the coordinates: $(\underset{x_1}{0}, \underset{y_1}{-2})$ $(\underset{x_2}{1}, \underset{y_2}{1})$

Step 3 – put into the formula:

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

$$m = \frac{1--2}{1-0}$$

$$= \frac{3}{1}$$

Answer: $m = 3$

The Equation of a Straight Line

Definition: the ***y*-intercept** of a straight line is the number on the y-axis that the line passes through.

This formula is <u>NOT</u> given on the exam paper
Equation of a straight line: $y = mx + c$

Where m is the gradient of the line and c is the y-intercept of the line.

In everyday language, this means that:

- the gradient is "the number before x"
- the y-intercept is "the number on its own"

<u>Examples</u>

Equation	Gradient	*y*-intercept
$y = 2x - 5$	2	–5
$y = 8 - x$	–1	8
$y = 4 - 3x$	–3	4
$y = 3 + \frac{5}{2}x$	$\frac{5}{2}$	3

The equation of the line must begin $y = ...$ If it does not, it must be rearranged:

Equation	Rearranged Equation	Gradient	*y*-intercept
$3y = 6x - 9$	$y = 2x - 3$	2	–3
$x + y = 5$	$y = 5 - x$	–1	5

Sketching a Straight Line from its equation

You need to know how to draw a line when given its equation. At Intermediate 1, you used a table of values to draw a straight line. You can still do this. However there is a quicker way that involves a knowledge of $y = mx + c$.

<u>Example 1 (Basic)</u>

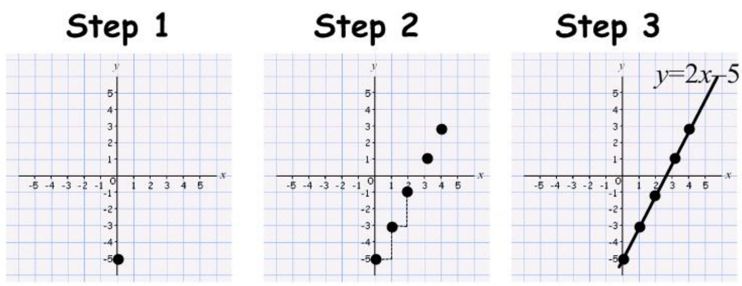
Draw the line $y = 2x - 5$

<u>Step 1</u> – the y-intercept is –5, so the line goes through (0,–5). Plot this point.

<u>Step 2</u> – the gradient is 2, so move 1 along and 2 up. Repeat this.

<u>Step 3</u> – draw and label the line.

(see diagrams on the next page)



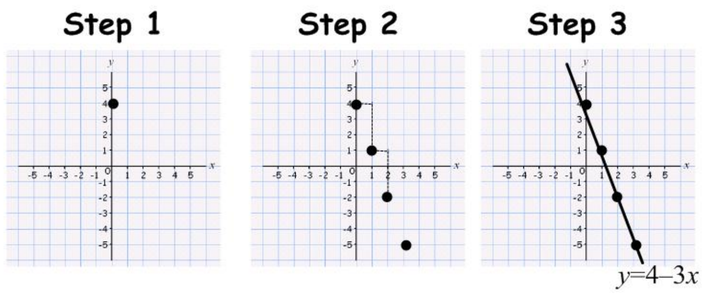
Example 2 (Negative Gradient)

Draw the line $y = 4 - 3x$

Step 1 – the y-intercept is 4, so the line goes through (0,4). Plot this point.

Step 2 – the gradient is –3. This is negative so the line slopes downwards. So from the first point, move 1 along and 3 down. Repeat this.

Step 3 – draw and label the line.



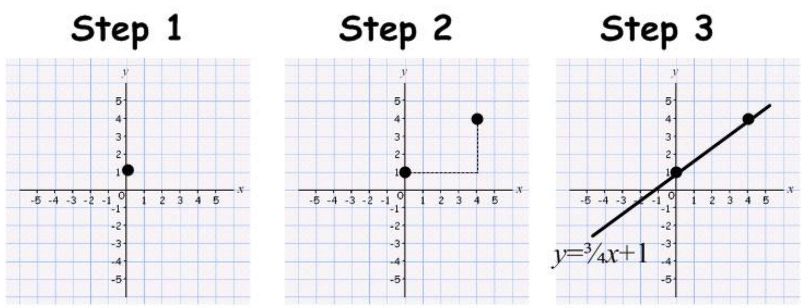
Example 3 (Fraction Gradient)

Draw the line $y = \frac{3}{4}x + 1$

Step 1 – the y-intercept is 1, so the line goes through (0,1). Plot this point.

Step 2 – the gradient is $\frac{3}{4}$. This means that you go **4 along** and **3 up** from your first point. Repeat this.

Step 3 – draw and label the line.

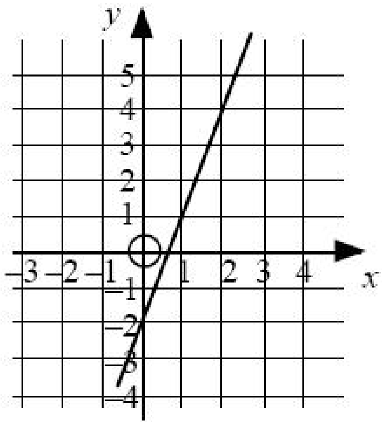


Writing down the equation of a graph from a diagram

Example 1 (Basic)

Find the equation of the straight line in this diagram.

IMPORTANT – show your working to get all the marks. Do not rush straight to step 3.



Step 1 – write down the y-intercept: $c = -2$

Step 2 – work out the gradient.

Two points on the line are (0,–2) and (1,1).

$m = \dfrac{y_2 - y_1}{x_2 - x_1} = \dfrac{1 - -2}{1 - 0} = \dfrac{3}{1} = 3$ *(you may be able to spot this from the diagram)*

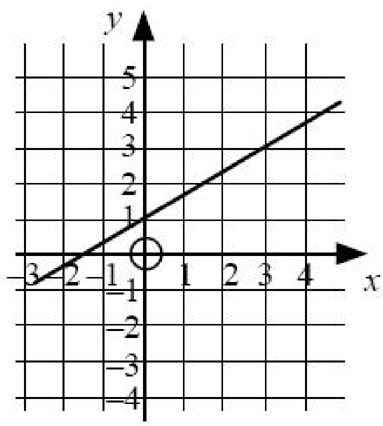
Step 3 – put these values into $y = mx + c$

Answer: $y = 3x - 2$

Example 2 (Harder)

Find the equation of the straight line in this diagram.

IMPORTANT – show your working to get all the marks. Do not rush straight to step 3.



Step 1 – write down the y-intercept.

$c = 1$

Step 2 – work out the gradient.

Two points on the line are (0,1) and (3,3).

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{3 - 1}{3 - 0} = \frac{2}{3}$$

Step 3 – put these values into $y = mx + c$

Answer: $y = \tfrac{2}{3}x + 1$

Unit 1 Outcome 4 – Algebra (Double Brackets and Factorising)

Multiplying Brackets

To multiply out double brackets, you have to multiply every term in the first bracket by every term in the second bracket. **You always need to simplify your answers – be very careful with negative signs.**

Example 1 (Basic)

$$(x-7)(x-9) = x^2 - 7x - 9x + 63$$
$$= \underline{x^2 - 16x + 63}$$

$$(2x+3)(x-4) = 2x^2 - 8x + 3x - 12$$
$$= \underline{2x^2 - 5x - 12}$$

Example 2 (Squaring brackets)

$(x+3)^2$ is NOT $x^2 + 9$. Instead you have to rewrite $(x+3)^2$ as $(x+3)(x+3)$

$$(x+3)^2 = (x+3)(x+3)$$
$$= x^2 + 3x + 3x + 9$$
$$= \underline{x^2 + 6x + 9}$$

Example 3 (Exam style questions with three terms in one bracket)

In these examples, the basic method is still the same, however you will have to do more multiplications. These questions will usually involve a term in x^3.

$$(x+4)(3x^2 - 2x + 5) = 3x^3 - 2x^2 + 5x + 12x^2 - 8x + 20$$
$$= \underline{3x^3 + 10x^2 - 3x + 20}$$

Factorising

There are three methods you can use to factorise expressions at Intermediate 2:

1. Take a common factor outside of the brackets (the Intermediate 1 method)
2. Difference of two squares
3. Trinomials

You should always check your answer by multiplying out the brackets.

Difference of two Squares

This method is really basic, but is easily forgotten about. It is called **Difference of Two Squares**. You can spot it because:

- There are only **two** terms
- They are being taken away (a **difference**)
- Both terms are **squares** (letters (x^2, a^2,...), square numbers (25, 81,...) or both)

The method:
Step 1 – write down two brackets
Step 2 – put a + sign in one bracket and a – sign in the other. (it does not matter which way around they go).
Step 3 – identify what goes at the start and end of each bracket.

Example 1 –

Factorise $a^2 - b^2$
Step 1 – (   ) (   )
Step 2 – (+) (–)
Step 3 – to make a^2 you do $a \times a$, so a goes at the start of each bracket. To make b^2 you do $b \times b$, so b goes at the end of each bracket.
Answer: $(a+b)(a-b)$

Example 2 –

Factorise $4x^2 - 25$
Step 1 – (   ) (   )
Step 2 – (+) (–)
Step 3 – to make $4x^2$ you do $2x \times 2x$, so $2x$ goes at the start of each bracket. To make 25 you do 5×5, so 5 goes at the end of each bracket.
Answer: $(2x+5)(2x-5)$

Example 3 –

Factorise $2x^2 - 32$
2 and 32 are not square numbers, so we cannot use this method (yet).
However we can take out a common factor of 2, to give $2(x^2 - 16)$. We can use this method on $x^2 - 16$, so we can complete the factorising.
Answer: $2(x+4)(x-4)$

Factorising Quadratic Expressions

A quadratic expression is one that contains a term in x^2. Examples include $x^2 + 4x + 3$, $x^2 - 4x - 5$ or $3x^2 + 7x - 6$. These factorise into **two brackets**.

Key fact:

- **The numbers in the brackets multiply to give the number at the end – e.g.:**
- For $x^2 + 4x + 3$, the two numbers in the bracket will multiply to give 3.
- For $3x^2 + 7x - 6$, the two numbers in the bracket will multiply to give –6.

Example 1 (easier – where it is just x^2 at the front)

Factorise $x^2 - x - 12$

Step 1 – make a list of all possible pairs of numbers

The two numbers in the bracket will multiply to give **–12**. So possibilities are:

–12 and +1,	–1 and +12,
–6 and +2,	–2 and +6,
–3 and +4	–4 and +3.

Step 2 – To make x^2, you do $x \times x$, so put x and x at the start of each bracket.

$$(x \quad)(x \quad)$$

Step 3 – experiment – try different pairs of numbers in the bracket. Multiply the brackets out to see if you can get $x^2 - x - 12$.

e.g. first you might try $(x-12)(x+1)$. But this multiplies out to give $x^2 - 11x - 12$, so this is NOT the answer.

e.g. next you might try $(x+4)(x-3)$. But this multiplies out to give $x^2 + x - 12$, so this is NOT the answer – but it is close.

e.g. finally you might try $(x-4)(x+3)$. This multiplies out to give $x^2 - x - 12$, which is want we want, so this is the answer.

Answer: $\underline{(x-4)(x+3)}$

Example 2 (harder – where there is a number before x^2)

Factorise $3x^2 + 11x + 6$

Step 1 – make a list of all possible pairs of numbers – **order matters**.

The two numbers in the bracket will multiply to give **+6**. So possibilities are:

+6 and +1,	+1 and +6,
+2 and +3	+3 and +2.

(technically we should also consider –6 and –1; –3 and –2 etc. As these also multiply to give +6. However in this question we can ignore negative numbers as there are no negative signs)

Step 2 – To make x^2, you do $3x \times x$, so put $3x$ and x at the start of each bracket.

$$(3x \quad)(x \quad)$$

Step 3 – experiment – try different pairs of numbers in the bracket. Multiply the brackets out to see if you can get $3x^2 + 11x + 6$.

e.g. first you might try $(3x+6)(x+1)$. But this multiplies out to give $3x^2 + 9x + 6$, so this is NOT the answer.

e.g. next you might try $(3x+2)(x+3)$. This multiplies out to give $3x^2 + 11x + 6$, which is want we want, so this is the answer.

Answer: $(3x+2)(x+3)$

Unit 1 Outcome 5 – Circles

Arc Length and Sector Area

These formulae are NOT given on the exam paper

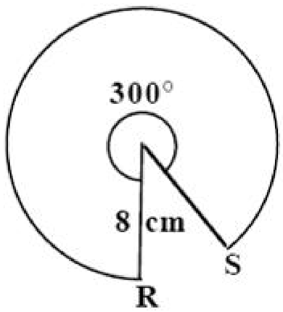
Arc length in a circle: Arc length $= \frac{\text{Angle}}{360}\pi d$

Sector area of a circle: Sector Area $= \frac{\text{Angle}}{360}\pi r^2$

The key idea in these questions is to identify the fraction of the circle that is in the question. This depends on the **angle** at the centre of the circle. This fraction is always $\frac{\text{Angle}}{360}$.

Example 1 (Arc Length)

Find the length of the arc in this sector of a circle

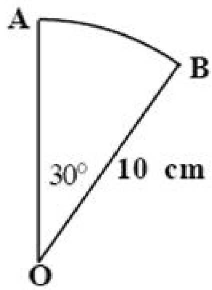


Radius is 8cm so diameter is 16cm.

$$\begin{aligned}\text{Arc length} &= \tfrac{300}{360}\pi d \\ &= \pi \times 16 \div 360 \times 300 \\ &= 41.88790205... \\ &= \underline{41.9\text{cm (1d.p.)}}\end{aligned}$$

Example 2 (Sector area))

Calculate the area of sector AOB in this diagram.



$$\begin{aligned}A &= \tfrac{30}{360}\pi r^2 \\ &= \pi \times 10 \times 10 \div 360 \times 30 \\ &= 26.17993878... \\ &= \underline{26.2\text{cm}^2\ \text{(1d.p.)}}\end{aligned}$$

Angles and Circles

Definition: a **tangent** to a circle is a line that just touches the edge of the circle.
Definition: a **chord** in a circle is a straight line going from one side of the circle to another.

Questions about angles and circles are all to do with identifying the right-angles, and then using the rules of angles to find any remaining angles.

You can only write a right angle in a diagram if you know it is a right angle. There are only three occasions when you can know an angle in a circle is a right-angle:

1. A tangent always makes a right angle with the radius.
2. Triangles in semi-circles are always right angled.

3. If a line in a circle diagram is a line of symmetry, it will cross any other line at right-angles.

Once you have identified the right angles, you have to use the other rules of angles to identify the other angles:

- Two angles on a straight line add up to make 180°.
- Opposite angles in X-shapes are equal.
- The three angles in a triangle add up to make 180°.
- The four angles in a quadrilateral add up to make 360°.
- Angles in Z shapes (made by parallel lines) are the same.
- Isosceles triangles are symmetrical, meaning that the two angles in the base are the same size. You can identify isosceles triangles in circle diagrams because two of their sides are likely to be the radius of the circle.

Pythagoras and SOH CAH TOA in triangles inside circles

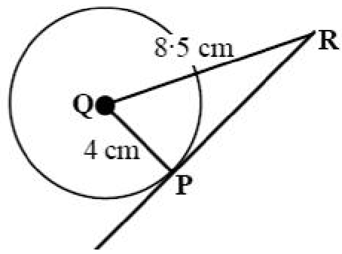
There are only three occasions when you can *know* an angle in a circle is a right-angle:

- A **tangent** always makes a right angle with the radius.
- Triangles in **semi-circles** are always right angled.
- If a line in a circle diagram is a **line of symmetry**, it will cross any other line at right-angles (and will split that line in two equal parts).

If there is a right-angle inside a triangle in a diagram, then you can use Pythagoras or trigonometry to work out other lengths or angles inside that triangle.

Example 1

PR is a tangent to a circle, centre Q.
Calculate angle PQR.



Solution

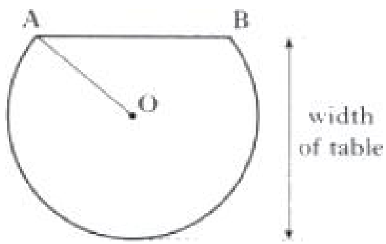
The fact that PR is a tangent means that the angle at P is 90°. This means that PQR is a right-angled triangle with the hypotenuse 8.5cm and the adjacent side 4cm.

$$\cos PQR = \tfrac{4}{8.5}$$
$$PQR = \cos^{-1}(4 \div 8.5)$$
$$PQR = \underline{61.9°}$$

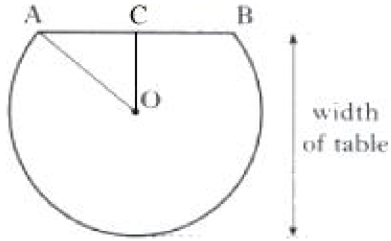
Example 2 (where a diagram is symmetrical)

A table top is in the shape of part of a circle.
The centre of the circle is O.
AB is a chord of the circle.
AB is 70cm.
The radius OA is 40cm.

Find the width of the table.



Solution



Step one – draw a new line in the diagram to make the diagram symmetrical.

In the diagram on the right, this is OC.
This means that triangle OCA is right-angled.

Step two – write in the length of the radius.

OA is 40cm.

Step three – use the fact that the new line splits the other line in half.

In this diagram, AB is 70cm, so AC must be 35cm.

Step four – use Trigonometry or Pythagoras to calculate another length (or angle) in the triangle.

In this diagram, triangle AOC is right angled. AO is 40cm and AC is 35cm.

Using Pythagoras:

$$\begin{aligned} OC^2 &= OA^2 - AC^2 \\ &= 40^2 - 35^2 \\ &= 375 \\ OC &= \sqrt{375} \\ &= \underline{19.4\text{cm (1d.p.)}} \end{aligned}$$

Step five – check whether you have answered the whole question

In this question, they wanted the width of the table. The width of the table is OC + the radius. So in this diagram, the width is 19.4+40=59.4cm.

Unit 2

Unit 2 Outcome 1 - Trigonometry

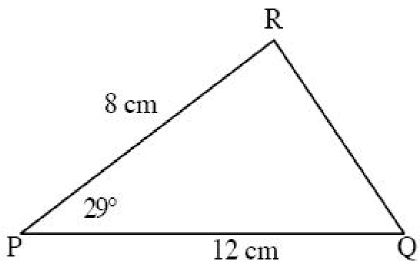
Area of a triangle

This formula is given on the exam paper
Area of a Triangle: $A = \frac{1}{2}ab\sin C$

To find the area of any triangle you need the length of two sides (a and b – it does not matter which one is which) and the size of the angle between them (C).

Example

Find the area of this triangle. Round your answer to 3 significant figures.



Solution

A=8, b=12 and C=29°

$$A = \frac{1}{2}ab\sin C$$

$$A = 8\times 12\times \sin 29 \div 2$$

$$A = 23.27086177...$$

$$\underline{A = 23.3\text{cm}^2 \ \text{(3s.f.)}}$$

Sine Rule

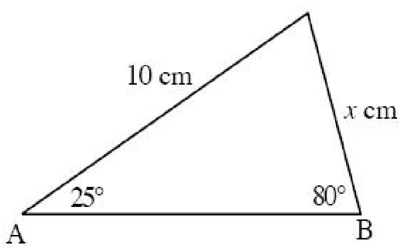
This formula is given on the exam paper
Sine rule: $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$

Where a, b and c are the lengths of the sides of the triangle, and A, B and C are the angles in the triangle. Side a is opposite angle A etc.

Important: to answer a question you do not use the formula as it is written. You only need the first two bits: $\frac{a}{\sin A} = \frac{b}{\sin B}$

Example

Find the length x in this triangle.



Solution

$x$cm is opposite 25°, so a=x and A=25
10cm is opposite 80°, so b=10 and B=80

$$\frac{a}{\sin A} = \frac{b}{\sin B}$$
$$\frac{x}{\sin 25} = \frac{10}{\sin 80}$$
$$x = \frac{10\sin 25}{\sin 80}$$
$$x = 4.291378296...$$
$$\underline{x = 4.29\text{cm (2d.p.)}}$$

The Cosine Rule

The formulae are <u>given</u> on the exam paper

Cosine rule: $a^2 = b^2 + c^2 - 2bc\cos A$ or $\cos A = \dfrac{b^2 + c^2 - a^2}{2bc}$

where a is the length you are finding, b and c are the other two sides, and A is the angle <u>between</u> the two sides b and c.

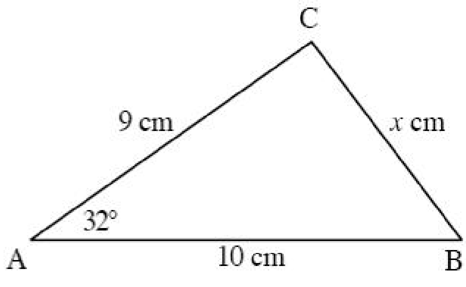
You use the first version of the formula to calculate a **length**, and the second version to calculate an **angle**.

Finding a length using the cosine rule

You <u>must know</u> the other two sides and the angle in between. It does not matter which side is called b and which is called c.

<u>Example</u>

Find the length of *x* in this diagram



Solution

It does not matter which side is b and which is c, so we will say that b=9 and c=10. A has to be the angle, so A is 32.

$$a^2 = b^2 + c^2 - 2bc\cos A$$
$$x^2 = 9^2 + 10^2 - 2\times 9\times 10\times \cos 32$$
$$x^2 = 81 + 100 - 180\cos 32$$
$$x^2 = 28.35134269....$$
$$x = \sqrt{28.35134269...}$$
$$x = 5.324549789$$
$$x = \underline{5.32\text{cm}}\ \text{(2d.p.)}$$

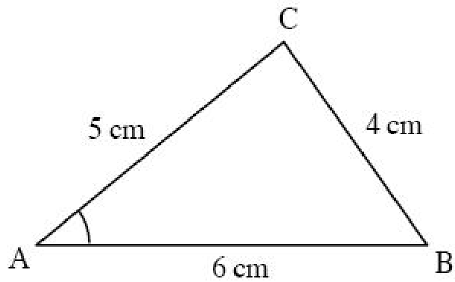
Finding an angle using the cosine rule

You <u>must know</u> the lengths of all three sides to be able to use this formula. To find an angle, you use the second version of the formula.

In the formula $\cos A = \dfrac{b^2 + c^2 - a^2}{2bc}$, *a* must be the side opposite the angle you are finding. It does not matter which way around *b* and *c* go.

Example

Find the size of angle CAB in this diagram



Solution

The side opposite the angle we are finding is 4cm, so *a* has to be 4. It does not matter which way around *b* and *c* go, so we will say *b*=5 and *c*=6.

$$\cos A = \frac{b^2 + c^2 - a^2}{2bc}$$

$$\cos A = \frac{5^2 + 6^2 - 4^2}{2 \times 5 \times 6}$$

$$\cos A = \frac{45}{60}$$

$$\cos A = 0.75$$

$$A = \cos^{-1} 0.75$$

$$A = 41.40962211....$$

$$A = \underline{41.4^\circ} \text{ (1d.p.)}$$

Unit 2 Outcome 2 – Simultaneous Linear Equations

Definition: a **simultaneous equation** is where you have more than one equation and more than one unknown letter, and where you have to find values for the letters that fit both equations.

Solving simultaneous equations algebraically

Example 1

Solve algebraically the system of equations $\begin{aligned}\mathbf{3x-2y=11}\\ \mathbf{2x+5y=1}\end{aligned}$

Solution

Step One – Multiply through each equation by the number at the start of the other equation

e.g. in this example, we multiply the top equation by 2, and the bottom equation by 3

$$3x-2y=11_{\times 2}$$
$$2x+5y=1_{\times 3}$$

$$6x-4y=22$$
$$6x+15y=3$$

Step Two – Take away the two equations to eliminate the x terms

$$\begin{array}{rrr} \cancel{6x} & -4y & =22 \\ -\cancel{6x} & +15y & =3 \\ \hline & -19y & =19 \end{array}$$

Step Three – solve the resulting equation: $-19y = 19$, so $y = -1$

Step Four – substitute this value for y back into one of the original equations

(either one will do). We will use the top one

$$3x-2y=11$$
$$3x-2\times-1=11$$
$$3x+2=11$$
$$3x=9$$
$$x=3$$

Answer: $\underline{x=3,\quad y=-1}$

Step Five – check your answer by substituting into the second equation.

Example 2

3 pencils and 2 books cost £10.30.

2 pencils and 3 books cost £15.20.

a) Write down a pair of equations to represent this situation

b) Solve these equations algebraically to find the cost of one book and one pencil.

Solution

a) The equations are $\begin{aligned} 3p + 2b &= 10.30 \\ 2p + 3b &= 15.20 \end{aligned}$

$$\begin{aligned} 3p + 2b &= 10.30_{\times 2} \\ 2p + 3b &= 15.20_{\times 3} \end{aligned}$$

b) Following the same steps as in the last example:

$$\begin{aligned} 6p + 4b &= 20.60 \\ 6p + 9b &= 45.60 \end{aligned}$$

$$\begin{array}{rrl} \cancel{6p} & +4b & = 20.60 \\ -\cancel{6p} & +9b & = 45.60 \\ \hline & 5b & = 25 \\ & b & = 5 \end{array}$$

Substituting back into top equation:

$$\begin{aligned} 3p + 2 \times 5 &= 10.30 \\ 3p + 10 &= 10.30 \\ 3p &= 0.30 \\ p &= 0.10 \end{aligned}$$

Answer: b=5 and p=0.10

<u>However you have to answer the question ina sentence to get the final mark – i.e. a book is £5 and a pencil is 10p.</u>

Solving simultaneous equations using a graph

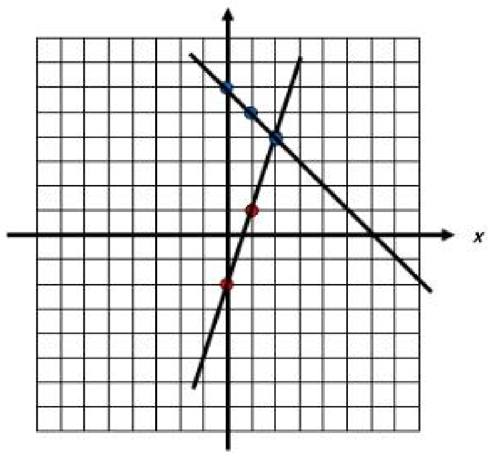
This method is much easier, but much less likely to come up in an exam.

The solution to two simultaneous equations is the point where the graphs of each equation cross each other. To find this, you need to be able to draw the graphs.

<u>Example</u>

Solve graphically the simultaneous equations $\begin{aligned} \boldsymbol{y = 3x - 2} \\ \boldsymbol{y = 6 - x} \end{aligned}$

Solution



<u>Step 1</u> – draw the two lines y=3x–2 and y=6–x using the method on page 9.

<u>Step 2</u> – write down the point where the two graphs cross

Answer: the graphs cross each other at the point (2,4), so the answer is <u>x=2, y=4</u>

Unit 2 Outcomes 3 and 4 – Graphs, Charts and Statistics

Pie Charts

You need to be able to work out the angles you would use to draw the slices in a pie chart. This is another example of a circles question – you have to work out what fraction of the circle each slice has to be.

Example

In a school survey 200 pupils were asked what their favourite takeaway food was. The results were:

Food	**Frequency**
Pizza	55
Fish and Chips	64
Chinese	81

How many degrees would you need for each sector to represent this data on a pie chart?

Solution

Step One – Work out the **fraction** for each slice

There were 200 pupils in total, so the fraction has to be out of 200

Pizza: $\frac{55}{200}$ Fish and chips: $\frac{64}{200}$ Chinese: $\frac{81}{200}$

Step Two – work out each slice as a fraction of 360°

Pizza: $\frac{55}{200}$ of $360° = \underline{99°}$

Fish and chips: $\frac{64}{200}$ of $360° = \underline{115.2°}$

Chinese: $\frac{81}{200}$ of $360° = \underline{145.8°}$

Step Three – check that your answer adds up to 360°

Scatter Graphs and Line of Best Fit

The line of best fit on a scattergraph is a straight line. This means that you can find the equation of a line of best fit using the method ($y=mx+c$) on page 9.

Once you have the equation, you can use the equation to estimate the value of y when you are told x (or vice versa). At Intermediate 2, you have to use the equation to get any marks (the question will say this). You *cannot* do it by "looking and guessing".

Example (2010 Exam Question)

A scattergraph shows the taxi fare *p* pounds plotted against the distance travelled, *m* miles. A line of best fit has been drawn.

The equation of the line of best fit is $p = 2 + 1.5m$. Use this equation to predict the taxi fare for a journey of 6 miles.

Solution

The journey is 6 miles, so m=6.

Using the equation,
$$\begin{aligned} p &= 2 + 1.5m \\ p &= 2 + 1.5 \times 6 \\ p &= 2 + 9 \\ p &= \underline{11\text{ miles}} \end{aligned}$$

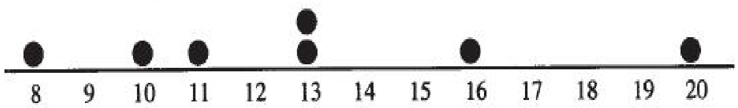
Dot Plots

A dot plot is an easy way of showing a list of numbers in a diagram.

Example

Construct a dot plot to show these numbers: **8, 10, 11, 13, 13, 16, 20**

Solution



Median, Quartiles and Box Plot

Definition: the **median** is the number that divides an ordered list of numbers into two equally-sized parts

Definition: the **quartiles**, along with the median, are the numbers that divide an ordered list into four equally-sized parts. The list must be written in order.

Definition: a **five-figure summary** of a list of numbers is the lowest (L), lower quartile (Q_1), median (Q_2), upper quartile (Q_3) and highest (H).

Definition: a **boxplot** is a way of showing and comparing five-figure summaries in a diagram.

Example

Draw a box plot from this set of numbers:
12, 14 , 15, 15, 16, 17, 17, 19, 19, 20, 21, 22, 26, 28, 31, 31

Solution

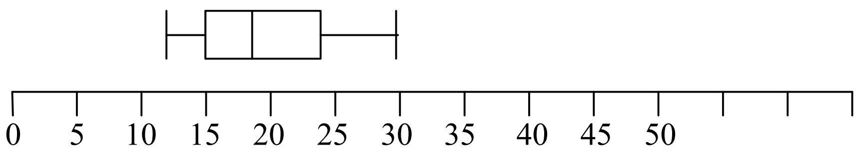
Step one – make a five-figure summary from the ordered list.

Lower Q Median Upper Q
L H
12, 14 , 15, 15, 16, 17, 17, 19, 19, 20, 21, 22, 26, 28, 31, 31

15.5 19 24

Lowest value	L= 12
Lower Quartile	$Q_1 = 15.5$
Median	$Q_2 = 19$
Upper Quartile	$Q_3 = 24$
Highest value	H = 31

Step two – draw the box plot, including an appropriate scale.



Semi-Interquartile Range

This formula is NOT given on the exam paper

Semi Interquartile-Range: $\text{SIQR} = \dfrac{\text{upper quartile} - \text{lower quartile}}{2}$

Example

For the data set above, work out the Semi Interquartile Range

Solution

Upper quartile = 24 and Lower quartile = 15.5, so $\text{SIQR} = \dfrac{24 - 15.5}{2} = \underline{4.25}$

Cumulative Frequency

Definition: the **cumulative frequency** is the running total of frequencies.

A cumulative frequency column can be added to a cumulative frequency table. All you have to do is add all the numbers together.

Example

Add a cumulative frequency column to this table

Age	Frequency
12	7
13	11
14	0
15	12

Solution

Age	Frequency	Cum. Frequency
12	7	7
13	11	18 *[7+11]*
14	0	18 *[7+11+0]*
15	12	30 *[7+11+0+12]*

Exam tip: If you are asked to find the median and quartiles from a cumulative frequency table, the best way is to write the list of data out in full first. E.g. in the example above write out **12, 12, 12, 12, 12, 12, 12, 13, 13, 13, 13, 13, 13, 13, 13, 13, 13, 13, 15, 15, 15, 15, 15, 15, 15, 15, 15, 15, 15, 15** and then proceed as normal.

Standard Deviation

Definition: the standard deviation of a list of numbers is a measure of how spread out the numbers are from the mean.

These formulae are <u>given</u> on the exam paper:

$$\text{standard deviation} = \sqrt{\frac{\sum (x-\bar{x})^2}{n-1}} = \sqrt{\frac{\sum x^2 - (\sum x)^2 / n}{n-1}}$$

Where n is how many numbers are in the list, $\bar{x}$ is the mean and Σ means "add together"

You only need to use one of these formulae. In general, it is more helpful to just know the method rather than memorising the formula.

Example

a) **Find the mean of these five numbers: 2, 3, 9, 6, 5**
b) **Find the standard deviation of the same five numbers**

Solution

a) $\frac{2+3+9+6+5}{5} = \frac{25}{5} = 5$, so the <u>mean is 5</u>

b) You have a choice of two methods:

Method 1 – first formula: $s = \sqrt{\frac{\sum (x-\bar{x})^2}{n-1}}$

<u>Step 1</u> - Draw up a table showing x, $x-\bar{x}$ and $(x-\bar{x})^2$

x	$x-\bar{x}$	$(x-\bar{x})^2$
2		
3		
9		
6		
5		

Step 2 – Complete the table, remembering that $\bar{x}$ = the mean $= 5$.

Step 3 – find the total of the final column

So $\sum(x-\bar{x})^2 = 30$

	x	$x-\bar{x}$	$(x-\bar{x})^2$
	2	–3	9
	3	–2	4
	9	4	16
	6	1	1
	5	0	0
TOTAL			**30**

Step 4 – use the formula, remembering that n=5 as there were five numbers.

$$s = \sqrt{\frac{\sum(x-\bar{x})^2}{n-1}}$$

$$= \sqrt{\frac{30}{5-1}}$$

$$= \sqrt{\frac{30}{4}} = 2.74 \text{ (2d.p.)}$$

Method 2 – second formula: $s = \sqrt{\dfrac{\sum x^2 - \frac{(\sum x)^2}{n}}{n-1}}$

Step 1 - Draw up a table showing x and x^2

x	x^2
2	
3	
9	
6	
5	

Step 2 – Complete the table

	x	x^2
	2	4
	3	9
	9	81
	6	36
	5	25
TOTAL	**25**	**155**

Step 3 – find the totals

So $\sum x = 25, \ \sum x^2 = 155$

Step 4 – use the formula, remembering that n=5.

$$s = \sqrt{\frac{\sum x^2 - \frac{(\sum x)^2}{n}}{n-1}}$$

$$= \sqrt{\frac{155 - \frac{25^2}{5}}{5-1}}$$

$$= \sqrt{\frac{155-125}{4}}$$

$$= \sqrt{\frac{30}{4}} = 2.74 \text{ (2d.p.)}$$

Probability

Probability is a measure of how likely an event is:

- If the event is **impossible**, the probability is 0
- If the event is **certain** to happen, the probability is 1
- If the event is between certain and uncertain, the probability is given as a fraction.

Example (2007 Exam Question)

The table shows the results of a survey of First Year pupils.

A pupil is picked at random from this sample, what is the probability that they are a girl wearing a blazer?

	Wearing a blazer	*Not wearing a blazer*
Boys	40	22
Girls	29	9

Solution

In total there are 40+29+22+9=100 pupils.
9 of these were girls wearing a blazer.
So the probability is $\frac{9}{100}$.

Comparing Statistics

The **mean**, **median** and **mode** are averages. They say whether a list of numbers is higher or lower on average.

The **range**, **semi-interquartile range** and **standard deviation** are measures of spread. They say whether a list of numbers is more or less spread out/consistent.

Example

The temperature in Aberdeen has a mean of 3°C and a standard deviation of 5. In London it has a mean of 9°C and a s.d. of 3. Compare the temperatures in London and Aberdeen.

Solution

You would get NO MARKS (as you are stating the obvious) for:
"Aberdeen has a lower mean", "London has a higher mean", "Aberdeen has a higher standard deviation", "London has a lower standard deviation".
You WOULD get marks (as you say what the numbers MEAN) for:
"The temperature in Aberdeen is lower and the temperature is less consistent"
"The temperature in London is higher and more consistent" or similar

Unit 3

Unit 3 Outcome 1 – Algebraic Operations

Simplifying Algebraic Fractions

You can simplify a fraction when there is a common factor (number *or* letter) on the top *and* on the bottom.

e.g. $\frac{9xy^2}{18x^2y} = \frac{\cancel{9}^{1}\,\cancel{x}\,y^{\cancel{2}}}{\cancel{18}_{2}\,x^{\cancel{2}}\,\cancel{y}} = \frac{y}{2x}$

In some examples, you may need to factorise before you can simplify:

Example 1

Simplify $\frac{v^2-1}{v-1}$

Solution

$$\frac{v^2-1}{v-1} = \frac{(v+1)\cancel{(v-1)}}{\cancel{(v-1)}} = \frac{v+1}{1} = v+1$$

Answer: $v+1$

Example 2

Simplify $\frac{2a^2+a-1}{2a^2+5a-3}$

Solution

$$\frac{2a^2+a-1}{2a^2+5a-3} = \frac{(2a-1)(a+1)}{(2a-1)(a+3)} = \frac{\cancel{(2a-1)}(a+1)}{\cancel{(2a-1)}(a+3)} = \frac{a+1}{a+3}$$

Answer: $\frac{a+1}{a+3}$

Multiplying Fractions

Multiplying fractions is the straightforward procedure – you **multiply the tops and multiply the bottoms**.

e.g. $\frac{3}{5}\times\frac{2}{7} = \frac{3\times 2}{5\times 7} = \frac{6}{35}$ $\qquad \frac{a}{c}\times\frac{b}{c} = \frac{ab}{c^2}$

It is easiest to cancel before you multiply. You are allowed to cancel *anything* from the top row with *anything* from the bottom row.

Example

Simplify $\frac{a^2}{15b} \times \frac{10}{a}$

Solution

Cancelling gives: $\frac{a^{\cancel{2}}}{\cancel{15}^{3}b} \times \frac{\cancel{10}^{2}}{\cancel{a}} = \frac{a}{3b} \times \frac{2}{1} = \frac{2a}{3b}$. **Answer**: $\underline{\frac{2a}{3b}}$

Dividing Fractions

To divide two fractions, you turn the second fraction upside down, and change the sum to be a multiply sum:

e.g. $\frac{3}{5} \div \frac{2}{7} = \frac{3}{5} \times \frac{7}{2} = \frac{21}{10}$ $\frac{x}{y} \div \frac{a}{x} = \frac{x}{y} \times \frac{x}{a} = \frac{x^2}{ay}$

Example

Divide and simplify: $\frac{6b}{ay} \div \frac{3ab}{x}$

Solution

Flip the second fraction upside down and multiply: $\frac{6b}{ay} \times \frac{x}{3ab}$

Cancel: $\frac{\cancel{6}^{2}\cancel{b}}{ay} \times \frac{x}{\cancel{3}_{1}a\cancel{b}} = \frac{2}{ay} \times \frac{x}{ab}$

Multiply: $\frac{2}{ay} \times \frac{x}{ab} = \frac{2x}{a^2by}$ **Answer:** $\underline{\frac{2x}{a^2by}}$

Adding and Subtracting Fractions

You can only add and subtract fractions when the denominators (the numbers on the bottom) are the same. When they are not the same, we can change the fractions into another fraction that *is* the same.

A quick method for doing this is:

- Step One – Multiply the two bottom numbers together to get the "new" denominator, which will be the same for each fraction.
- Step Two – Multiply diagonally to get the "new" numerators (top numbers) for each fraction
- Step Three – Add or take away the top line

Examples (Basic)

$\frac{2}{3}-\frac{3}{5}$		$\frac{a}{b}+\frac{2}{c}$
$\frac{}{15}-\frac{}{15}$	Step 1 – multiply the two bottom numbers together	$\frac{}{bc}+\frac{}{bc}$
$\frac{10}{15}-\frac{}{15}$	Step 2 – multiply diagonally: top-left and bottom-right. The answer goes in the top-left.	$\frac{ac}{bc}+\frac{}{bc}$
$\frac{10}{15}-\frac{9}{15}$	Step 3 – multiply diagonally: top-right and bottom-left. The answer goes in the top-right.	$\frac{ac}{bc}+\frac{2b}{bc}$
$\underline{\frac{1}{15}}$	Step 4 – do the adding or taking away in the top line	$\underline{\frac{ac+2b}{bc}}$

Example 2

Simplify $\frac{4x}{5}+\frac{3}{x}$

Solution

$$\frac{4x}{5}+\frac{3}{x}=\frac{4x^2}{5x}+\frac{15}{5x}=\frac{4x^2+15}{5x}.$$

Answer: $\underline{\frac{4x^2+15}{5x}}$

Example 3

Simplify $\frac{x}{x+2}+\frac{3}{x-4}$

When a fraction has more than one term on the top or bottom, you need to introduce **brackets**. You then deal with a bracket as a single object.

$$\begin{aligned}\frac{x}{x+2}+\frac{3}{x-4} &= \frac{x}{(x+2)}+\frac{3}{(x-4)}\\ &= \frac{x(x-4)}{(x+2)(x-4)}+\frac{3(x+2)}{(x+2)(x-4)}\\ &= \frac{x^2-4x}{(x+2)(x-4)}+\frac{3x+6}{(x+2)(x-4)}\\ &= \frac{x^2-4x+3x+6}{(x+2)(x-4)}\\ &= \frac{x^2-x+6}{(x+2)(x-4)}\end{aligned}$$

Answer: $\underline{\frac{x^2-x+6}{(x+2)(x-4)}}$

Changing the subject of a formula

Changing the subject of a formula is just like rearranging an equation: you move things from one side to the other and do the opposite.

A useful tip for changing the subject questions is to switch the left-hand side and the right-hand side before you begin moving things.

Example 1

Change the subject of the formula $y = ab + d$ to 'b'

Solution

Step One – flip the left and right hand sides: $ab + d = y$

Step Two – rearrange

$$ab = y - d$$

$$b = \frac{y-d}{a}$$

Example 2

Change the subject of the formula $A = \pi d^2$ to 'd'

Solution

Step One – flip the left and right hand sides: $\pi d^2 = A$

Step Two – rearrange

$$\pi d^2 = A$$

$$d^2 = \frac{A}{\pi}$$

$$d = \sqrt{\frac{A}{\pi}}$$

Simplifying Surds

Definition: a **surd** is a square root (or cube root etc.) which does not have an exact answer. e.g. $\sqrt{2} = 1.414213562...$, so $\sqrt{2}$ is a surd. However $\sqrt{9} = 3$ and $\sqrt[3]{64} = 4$, so $\sqrt{9}$ and $\sqrt[3]{64}$ are not surds because they have an exact answer. To simplify a surd, you need to look for square numbers that are factors of the original number.

Examples (Basic)

Express $\sqrt{75}$ and $\sqrt{98}$ in their simplest form

Solution

$$\begin{aligned}\sqrt{75} &= \sqrt{25 \times 3} \\ &= \sqrt{25} \times \sqrt{3} \\ &= 5 \times \sqrt{3} \\ &= \underline{5\sqrt{3}}\end{aligned}$$

$$\begin{aligned}\sqrt{98} &= \sqrt{49 \times 2} \\ &= \sqrt{49} \times \sqrt{2} \\ &= 7 \times \sqrt{2} \\ &= \underline{7\sqrt{2}}\end{aligned}$$

You can only add or take away surds when the number underneath the surd sign is the same.

e.g. Simplify $\sqrt{5}+\sqrt{3}$ is NOT $\sqrt{8}$. Instead the simplest answer is $\sqrt{5}+\sqrt{3}$ (i.e. no change), because no simplifying is possible.

Examples 2 (Harder)

Write as a surd in its simplest form: $\sqrt{63}+\sqrt{7}-\sqrt{28}$

Solution

Step One – simply all three surds

$$\begin{aligned} & \sqrt{63} && +\sqrt{7} && -\sqrt{28} \\ = & \sqrt{9\times 7} && +\sqrt{7} && -\sqrt{4\times 7} \\ = & \sqrt{9}\times\sqrt{7} && +\sqrt{7} && -\sqrt{4}\times\sqrt{7} \\ = & 3\sqrt{7} && +\sqrt{7} && -2\sqrt{7} \end{aligned}$$

Step Two – add and take away: $3\sqrt{7}+\sqrt{7}-2\sqrt{7}=\underline{2\sqrt{7}}$

Rationalising the Denominator

For various mathematical reasons, it is not good to have a surd on a bottom of a fraction.

Definition: Rationalising the denominator means turning the surd at the bottom of the fraction into a whole number, whilst keeping the fraction the same.

The method is very simple: **multiply top and bottom by the surd**.

Example 1

Express $\frac{4}{\sqrt{5}}$ **with a rational denominator**

Solution

Multiply top and bottom by $\sqrt{5}$: $\frac{4}{\sqrt{5}}\times\frac{\sqrt{5}}{\sqrt{5}}=\underline{\frac{4\sqrt{5}}{5}}$

Rules of Indices

Basic Rule 1: anything to the power 0 is equal to 1:

e.g. $5^0=1,\quad 17^0=1,\quad 35627658^0=1$

Basic Rule 2: anything to the power 1 is equal to itself:

e.g. $5^1=5,\quad 17^1=17,\quad 35627658^1=35627658$

Key Rule 1: when you multiplying two expressions involving powers, you add the numbers in the power: $a^m \times a^n = a^{m+n}$

e.g. $x^3 \times x^4 = \underline{x^7}$ $\quad y^{-1} \times y^6 = y^{-1+6} = \underline{y^5}$

Key Rule 2: when you divide two expressions involving powers, you take away the numbers in the power: $\frac{a^m}{a^n} = a^{m-n}$

e.g. $a^8 \div a^3 = \underline{a^5}$ $\quad \frac{m^{10}}{m^8} = \underline{m^2}$

Key Rule 3: when you one power to the power of another, you multiply the numbers in the power: $(a^m)^n = a^{mn}$

e.g. $(x^2)^3 = \underline{x^6}$ $\quad (a^4)^{-2} = \underline{a^{-8}}$

Negative Powers

A negative power is all to do with dividing. In general, $a^{-m} = \frac{1}{a^m}$

e.g. $3^{-2} = \frac{1}{3^2} = \frac{1}{\underline{\underline{9}}}$ $\quad a^{-4} = \frac{1}{\underline{\underline{a^4}}}$ $\quad 5x^{-2} = 5 \times \frac{1}{x^2} = \frac{5}{\underline{\underline{x^2}}}$

Example 1

Rewrite $3x^{-4}$ using positive powers

Solution

$\frac{3}{\underline{\underline{x^4}}}$

Fractions in Powers

A fraction as a power is to do with a root (square root, cube root etc.) of some form.

In general, $a^{\frac{m}{n}} = \sqrt[n]{a}^m$

e.g. $64^{\frac{1}{3}} = \sqrt[3]{64}^1 = 4^1 = \underline{4}$

$$\begin{aligned} 9^{\frac{3}{2}} &= \sqrt[2]{9}^3 \\ &= 3^3 \\ &= \underline{27} \end{aligned}$$

$x^{\frac{2}{5}} = \sqrt[5]{x}^2$

Exam Style Questions Involving Indices

You need to be able to handle questions using a mixture of these rules.

Example 1

Simplify $25^{-\frac{1}{2}}$

Solution

$$25^{-\frac{1}{2}} = \frac{1}{25^{\frac{1}{2}}} = \frac{1}{\sqrt{25}} = \frac{1}{\underline{\underline{5}}}$$

Example 2

Simplify $3x^2(x^{-2} + 2x^5)$

Solution

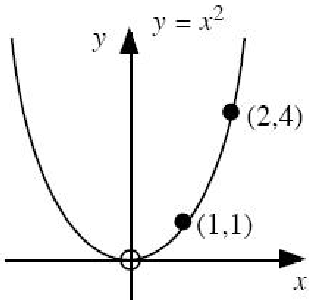
$$\begin{aligned} 3x^2(x^{-2} + 2x^5) &= 3x^2 \times x^{-2} + 3x^2 \times 2x^5 \\ &= 3x^{2+(-2)} + 6x^{2+5} \\ &= 3x^0 + 6x^7 \\ &= \underline{3 + 6x^7} \end{aligned}$$

Unit 3 Outcome 2 – Quadratic Functions

Definition: a quadratic function is one that contains a squared term and no higher powers. e.g. $3x^2$, x^2-4 and x^2+5x+1 are all quadratic functions, but x^5 and x^2+x^3 are not.

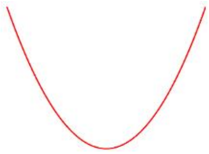
Graphs of Quadratic Functions

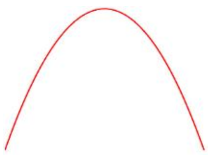
Definition: the graph of a quadratic function is called a **parabola**.



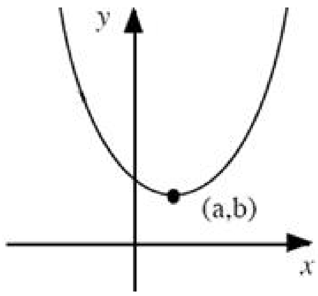
The graph on the right is the basic graph of $y = x^2$. You should know its shape.

If x^2 is positive , then the graph is "happy" (has a minimum turning point)





If x^2 is negative, then the graph is "unhappy" (has a maximum turning point)



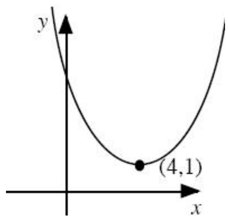
The graph of $y=(x-a)^2+b$ is still a parabola, but it looks like this:

Key facts:
(You would normally be given these in an exam question, but you might not be.)

- $y=(x-a)^2+b$ is a "happy" parabola. The minimum point of $y=(x-a)^2+b$ is (a,b)
- $y=-(x-a)^2+b$ is an "unhappy" parabola. The maximum point of $y=-(x-a)^2+b$ is (a,b)
- The **axis of symmetry** of $y=(x-a)^2+b$ has the equation $x=a$

Example

What is the equation of this parabola?



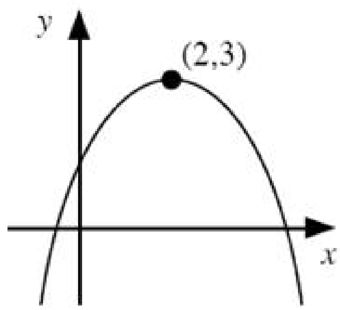
Solution

The graph is happy, so it has an equation of the form $y=(x-a)^2+b$

The minimum point is (4,1) so the equation is $\underline{y=(x-4)^2+1}$

Example 2

What is the equation of this parabola?



Solution

The graph is unhappy, so it has an equation of the form $y = -(x-a)^2 + b$

The maximum point is (2,3) so the equation is
$\underline{y = -(x-2)^2 + 3}$ (or $y = 3-(x-2)^2$).

Equation of a Graph from a Point on the Graph

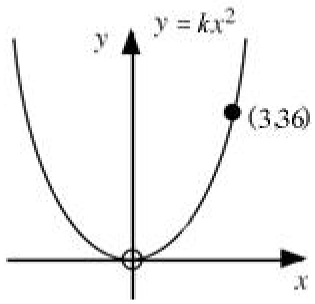
The coordinates of any point on a graph tells you a value for x and y.

e.g. for the coordinate point (3 , 7), $x = 3$ and $y = 7$
e.g. for the coordinate point (0 , 5), $x = 0$ and $y = 5$
e.g. for the coordinate point (–4 , 1), $x = -4$ and $y = 1$

These values can be put back into the equation of the graph. If you don't know the full equation of a graph, they can give you an equation to solve to complete it.

Example

The graph on the right has the equation $y = kx^2$. The graph passes through the point (3, 36). Find the value of k.



Solution

A point on the graph is (3 , 36). This means that $x = 3$ and $y = 36$.

Substituting these values into the equation gives:

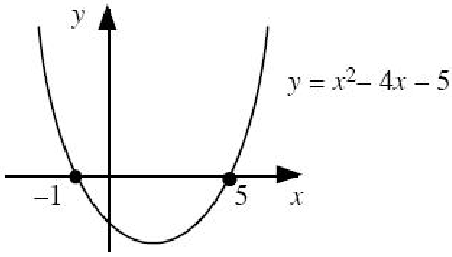
$$y = kx^2$$
$$36 = k \times 3^2$$
$$36 = k \times 9$$
$$\underline{k = 4}$$

Solving Quadratic Equations from a graph

Definition: the **roots** of an equation are another word for its solutions.
The roots of a quadratic equation are the points that the parabola crosses the x-axis.

Example

Using the graph shown, write down the two solutions of the equation $x^2 - 4x - 5 = 0$



Solution

The roots are $x = -1$ and $x = 5$.

Solving Quadratic Equations by factorising

Factorising is the simplest way of solving a quadratic equation, but you can only use it when the expression can actually be factorised!

Important – you must rearrange the equation so that is has =0 on the right-hand side. If you do not do this, you will risk losing all of the marks.

Example 1

Use factorising to solve the equation $2x^2 - 6x = 0$

Solution

Step 1 – check that the equation has '=0' on the right-hand side.
On this occasion, it does, so we do not need to do anything more.

Step 2 – factorise the expression

$$2x^2 - 6x = 0$$
$$2x(x-3) = 0$$

Step 3 – split up into two separate equations and solve

$$2x = 0 \qquad x - 3 = 0$$
$$x = 0, \qquad x = 3$$

Example 2 (Difference of Two Squares)

Use factorising to solve the equation $y^2 - 49 = 0$

Solution

Step 1 – check that the equation has '=0' on the right-hand side.
On this occasion, it does, so we do not need to do anything more.

Step 2 – factorise the expression

$$y^2 - 49 = 0$$
$$(y+7)(y-7) = 0$$

Step 3 – split up into two separate equations and solve

$$y + 7 = 0 \qquad y - 7 = 0$$
$$y = -7, \qquad y = 7$$

Example 3 (Number in front of x^2)

Use factorising to solve the equation $2x^2 + 9x - 5 = 0$

Solution

Step 1 – check that the equation has '=0' on the right-hand side.
On this occasion it does, so we do not need to do anything more.

Step 2 – factorise the expression

$$2x^2 + 9x - 5 = 0$$
$$(2x-1)(x+5) = 0$$

Step 3 – split up into two separate equations and solve

$$2x - 1 = 0 \qquad x + 5 = 0$$
$$2x = 1$$
$$\underline{x = \tfrac{1}{2}, \qquad x = -5}$$

Example 4 (Right-hand side not zero)

Use factorising to solve the equation $x^2 - 2x - 10 = 5$

Solution

Step 1 – check that the equation has '=0' on the right-hand side.

$$x^2 - 2x - 10 = 5$$

It does not, so we need to rearrange: $x^2 - 2x - 10 - 5 = 0$

$$x^2 - 2x - 15 = 0$$

Step 2 – factorise the rearranged expression

$$x^2 - 2x - 15 = 0$$
$$(x+3)(x-5) = 0$$

Step 3 – split up into two separate equations and solve

$$x + 3 = 0 \qquad x - 5 = 0$$
$$\underline{x = -3, \qquad x = 5}$$

The Quadratic Formula

This formula is given on the exam paper

The roots of $ax^2 + bx + c = 0$ are given by: $$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

The quadratic formula can be used to solve *any* quadratic equation. We usually use it when you can't factorise the expression.

Important – you must rearrange the equation so that is has =0 on the right-hand side. If you do not do this, you will risk losing all of the marks.

A clue to use the formula is where the questions tells you to "**give your answers correct to 2 decimal places**"

Example 1

Solve the equation $3x^2 + 2x - 6 = 0$**, giving your answers correct to two decimal places.**

Solution

Step 1 – check the equation has '=0' on the right. It does, so we can proceed.
Step 2 – write down what *a*, *b* and *c* are.

$$a = 3,\ b = 2,\ c = -6$$

Step 3 – substitute into the formula and solve – **being very careful when dealing with negative signs**:

$$x = \frac{-2 \pm \sqrt{2^2 - 4 \times 3 \times (-6)}}{2 \times 3}$$
$$x = \frac{-2 \pm \sqrt{4 - (-72)}}{6}$$
$$x = \frac{-2 \pm \sqrt{76}}{6}$$

$$x = \frac{(-2 + \sqrt{76})}{6} \qquad x = \frac{(-2 - \sqrt{76})}{6}$$
$$x = 1.12 \text{ (2d.p.)} \qquad x = -1.79 \text{ (2d.p.)}$$

If the number under the square root sign works out to be negative, then you will not be able to complete the formula. This means either that:

- You have made a mistake with negative numbers and need to check your working (realistically this is the most likely thing that would have happened in an exam)
- Or the equation has no solution (happens a lot in real life, but less likely in an exam)

Example 2

Solve the equation $2x^2 - 5x - 1 = 3$**, giving your answers correct to 2d.p.**

Solution

Step 1 – check the equation has '=0' on the right-hand side. It does not, so we have to rearrange:

$$2x^2 - 5x - 1 - 3 = 0$$
$$2x^2 - 5x - 4 = 0$$

Step 2 – write down what *a*, *b* and *c* are.

$$a = 2,\ b = -5,\ c = -4$$

Step 3 – substitute into the formula and solve – **being very careful when dealing with negative signs**:

$$x = \frac{-(-5) \pm \sqrt{(-5)^2 - 4 \times 2 \times (-4)}}{2 \times 2}$$
$$x = \frac{5 \pm \sqrt{25 - (-32)}}{4}$$
$$x = \frac{5 \pm \sqrt{57}}{4}$$

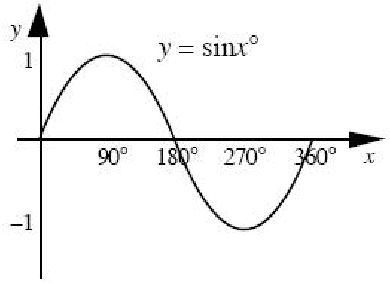
$$x = \frac{(5 + \sqrt{57})}{4} \qquad\qquad x = \frac{(5 - \sqrt{57})}{4}$$
$$x = 3.14 \text{ (2d.p.)} \qquad\qquad x = -0.64 \text{ (2d.p.)}$$

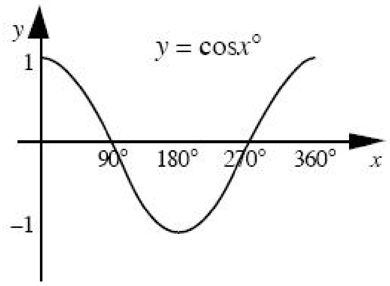
Answer: $x = 3.14$ and $x = -0.64$

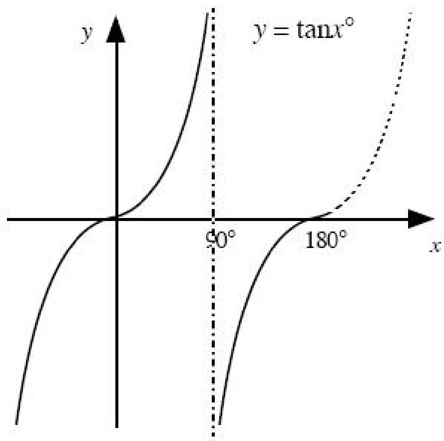
Unit 3 Outcome 3 – Further Trigonometry

Graphs of sin, cos and tan

You are should know what the graphs of sin x, cos x and tan x look like between 0° and 360°:







Definition: the **period** of a graph is how many degrees it takes the graph to do one complete cycle. In the graphs above, sin x and cos x have a period of 360° and tan x has a period of 180°.

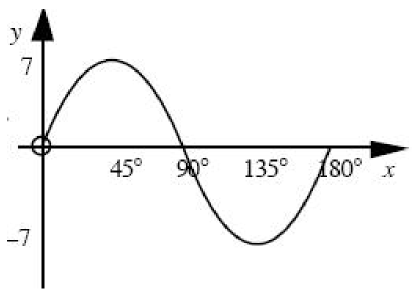
Definition: the **frequency** of a sin or cos graph is how many times the graph repeats itself in 360°. The frequency of a tan graph is how many times it repeats itself in 180°. In the equation of a sin, cos or tan graph, the frequency is the number before x.

Definition: the **amplitude** is a way of describing the height of a sin or cos graph – e.g. the sine and cosine graphs above both have an amplitude of 1. In the equation of a sin or cos graph, the amplitude is the number before sin or cos.

Equation	**Frequency**	**Amplitude**
$y = \cos x$	1	1
$y = 3\sin 4x$	4	3
$y = 5\cos 2x$	2	6

Example 1

The graph on the right has an equation of the form $y = a\sin bx$. What are the values of a and b?



Solution

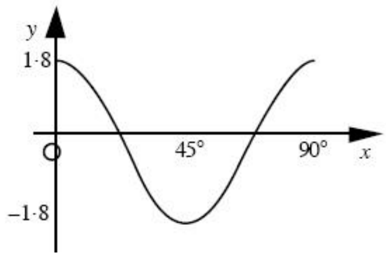
The maximum and minimum are 7 and –7, so the amplitude is 7 – i.e. a=7.

The graph repeats itself once between 0° and 180°. This means that it repeats itself twice between 0° and 360°, so it has a frequency of 2, so b=2.

Answer: the graph is $\underline{y = 7\sin 2x}$

Example 2

The graph on the right has an equation of the form $y = a\cos bx$. What are the values of a and b?



Solution

The maximum and minimum are 1.8 and –1.8, so the amplitude is 1.8. This means that a=1.8.

The graph repeats itself once between 0° and 90°. This means that it repeats itself four times between 0° and 360°, so it has a frequency of 4, meaning b=4.

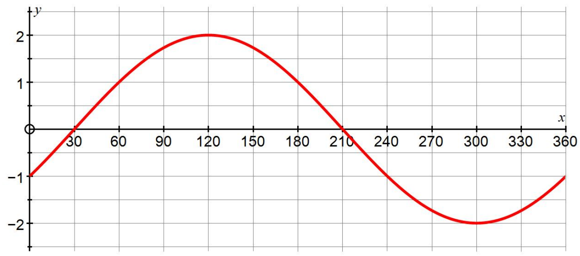
Answer: the graph is $\underline{y = 1.8\sin 4x}$

Definition: the **phase angle** is the amount a graph has been shifted to the right. In the equation of a sin, cos or tan graph, the phase angle is the number taken away from x in the brackets.

Equation	Period	Amplitude	Phase Angle
$y = 2\cos(x-45)°$	0	2	45°
$y = \sin(x-30)°$	1	1	30°
$y = 4\cos(2x-15)°$	2	4	15°

Example 3

The graph on the right has an equation of the form $y = a\sin(x-b)°$. What are the values of a and b?



Solution

The maximum and minimum are 2 and –2, so the amplitude is 2. So this means that a=2.

The graph has been shifted 30° to the right, so b=45°.

Answer: the graph is $\underline{y = 2\sin(x-45)°}$

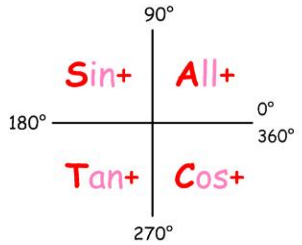
Solving Trigonometric Equations (Equations with sin, cos and tan in)

You can recognise these questions because they will ask you to **solve each equation for $0 \le x < 360$.** (This just means x must between 0° and 360°.)

These questions will usually have TWO answers:

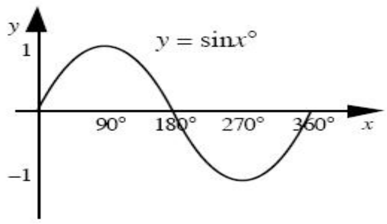
- Your calculator will give you the first using $\sin^{-1}$, $\cos^{-1}$ or $\tan^{-1}$
- To get the other, you need either a CAST diagram or a sketch of the graph.

$180 - x$



$180 + x$ $360 - x$

or



Example 1 (Positive values of sin, cos and tan)

Solve the equation $5\sin x - 2 = 1$ for $0 \le x < 360$:

Solution:

Step One – rearrange the equation

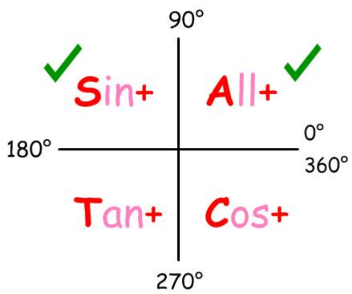
$$5\sin x - 2 = 1$$
$$5\sin x = 1 + 2$$
$$5\sin x = 3$$
$$\sin x = \tfrac{3}{5}$$

Step Two – find the first solution using $\sin^{-1}$

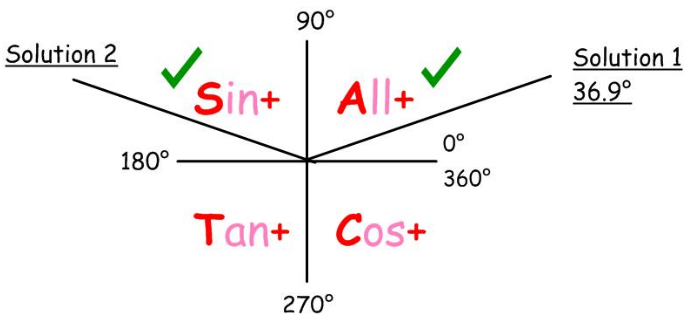
$$x = \sin^{-1}(3 \div 5)$$
$$x = 36.9°$$

Step Three – find the second solution using CAST

This question involves **sin**. The number on the right is $\tfrac{3}{5}$, which is **positive**. This means we tick the ALL and SIN quadrants.



Putting both answers into the diagram shows using symmetry that solution 2 is given by 180 – 36.9 = 143.1°.



Answer: x=36.9°, x=143.1°

Example 2 (Negative values of sin, cos and tan)

Solve the equation $3\cos x + 3 = 1$ for $0 \le x < 360$:

Solution:

Step One – rearrange the equation

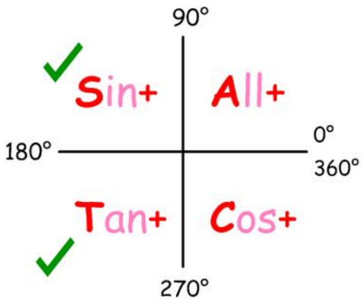
$$3\cos x + 3 = 1$$
$$3\cos x = 1 - 3$$
$$3\cos x = -2$$
$$\cos x = \tfrac{-2}{3}$$

Step Two – find the first solution using $\cos^{-1}$

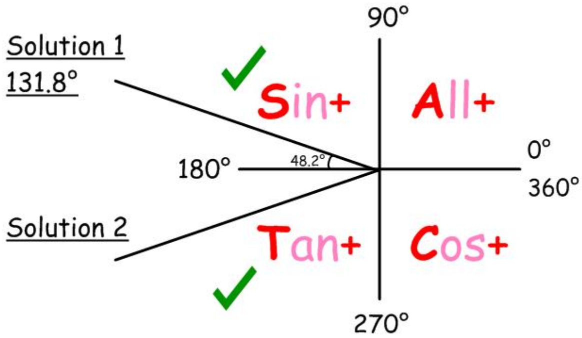
$$x = \cos^{-1}(-2 \div 3)$$
$$x = 131.8°$$

Step Three – find the second solution using CAST

This question involves **cos**. The number on the right is $\frac{-2}{3}$, which is **negative**. This means that cos is NOT positive, so we do NOT tick all and cos, instead we tick the SIN and TAN quadrants.



Putting both answers into the diagram shows using symmetry that solution 2 is given by 180 + 48.2 = 228.2°.



Answer: x=131.8°, x=228.2°

Trigonometric Identities

These formulae are NOT given on the exam paper

$$\tan x = \frac{\sin x}{\cos x} \qquad \sin^2 x + \cos^2 x = 1$$

We use trigonometric identities to simplify more complex expressions. A question would normally ask you to "Prove that..." a fact about sin, cos or tan is true or to "simplify" an expression involving sin, cos or tan.

There are not really any set rules for how to do these questions – instead you have to use your mathematical ability to choose the correct rules of algebra to find the answer. However the following tips are a good starting point:

Tip 1 – only ever "do stuff" to the more complicated expression (often the one on the left-hand side). Leave the simpler expression alone.

Tip 2 – you only get marks for knowing and using the two formulae in the grey box above. So a good bit of advice is to look at the more complicated expression, and:

a) If you see ' $\tan x$ ' on the left hand side, replace it with $\dfrac{\sin x}{\cos x}$

b) If you see ' $\dfrac{\sin x}{\cos x}$ ' on the left hand side, replace it with $\tan x$

c) If you see ' 1 ' on the left hand side, replace it with $\sin^2 x + \cos^2 x$

d) If you see ' $\sin^2 x + \cos^2 x$ ' on the left hand side, replace it with 1

Example 1

Prove that $\mathbf{\dfrac{1-\cos^2 x}{3\sin^2 x} = \dfrac{1}{3}}$

Solution

Using Tip 1: The left-hand side is more complicated, and the right-hand side is simpler. This means that we will "do stuff" to the left-hand side $\dfrac{1-\cos^2 x}{3\sin^2 x}$.

Using Tip 2: Using tip 2c above, we replace '1' with $\sin^2 x + \cos^2 x$:

$$\frac{1-\cos^2 x}{3\sin^2 x} = \frac{\sin^2 x + \cos^2 x - \cos^2 x}{3\sin^2 x}$$

We can now do some simplifying

$$\frac{\sin^2 x \cancel{+\cos^2 x} \cancel{-\cos^2 x}}{3\sin^2 x} = \frac{\sin^2 x}{3\sin^2 x} = \frac{{}^{1}\cancel{\sin^2 x}}{3\cancel{\sin^2 x}} = \frac{1}{3}$$, which is what we wanted

---

Example 2

Simplify $\mathbf{5\sin^2 x + 5\cos^2 x}$

Solution

Tip 1 isn't relevant here as there is only one expression.
Using **Tip 2**, none of those four expressions appear on the left-hand side exactly. However if we spot there is a common factor of 5 and <u>factorise it first</u>, we can use tip 2d:

$$\begin{aligned} 5\sin^2 x + 5\cos^2 x &= 5(\sin^2 x + \cos^2 x) \\ &= 5(1) \\ &= 5 \end{aligned}$$

. **Answer:** <u>5</u>

Index of Key Words